I shall cherish this memorial also for that it bears the likeness of one whose true scientific spirit we all learned to admire, and whom, for his genial character, we all learned to love.''

The achievements of Professor Newton, great as they were from a scientific standpoint, give no adequate idea, taken in themselves, of his power and influence. These, in a larger sense have become a part of the organic life of the University where his work was done. He built up, during a leadership of forty years, a strong and symmetrical department of Mathematics, by his comprehensive grasp of the trend of Mathematical thought, and by his wonderful power of divining the paths which lead out to fruitful fields of research, both within the domain of pure mathematics and in its applications to other sciences. Nor was the best part of his academic activities merely in his own department of studies. In moulding the general policy of the institution his counsel was invaluable; in establishing and maintaining the moral and intellectual standards, his influence was preëminent; the University bears the indelible impress of a life consecrated to the development of the noblest ideals.

Yale University.

ON THE SOLUTION OF THE QUADRATIC EQUATION.

By G. A. MILLER, Ph. D., Paris, France.

[Continued from January Number.]

The solution of the quadratic equation

$$a_0x^2+a_1x+a_2=0 \qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad A$$

is clearly equivalent to finding the two factors which are linear in x of the quantic

$$a_0x^2+a_1x+a_2.$$

When we ask whether this quantic has linear factors it is necessary to consider the domain of rationality to which we confine our attention. For illustration, we may consider the special quantic

$$x^2-4x+1.$$

If we confine ourselves to the simplest domain of rationality, viz: the domain which consists of all the rational numbers, we have to say that this quantic has no linear factors. In other words, it is irreducible in this domain. Howev-

er, if we enlarge this domain by adding to it the irrational number $\sqrt{3}$* we obtain a domain in which the quantic is clearly reducible. This domain is composed of all the numbers whose form is

$$\alpha+\beta\sqrt{3} \qquad (\alpha \text{ and } \beta \text{ being any rational numbers}).$$

According to the fundamental theorem of algebra a quantic which involves only a single variable can always be resolved into its linear factors in the domain obtained by enlarging the domain of its coefficients, if necessary, so as to include suitable new numbers. If the coefficients lie in the domain of the complex numbers the added numbers must also lie in this domain. If a quantic involves several variables it may remain irreducible even when the domain is enlarged in every possible manner.

Let x_1 and x_2 be the two roots of A. Since every rational symmetric function of the roots of an algebraic equation can be expressed rationallly in terms of its coefficients we know the value of any rational symmetric function of x_1 and x_2. This value must lie in the domain of the coefficients. In particular, we know the value of any even power of x_1-x_2. The value of the square is given by the equation

$$(x_1-x_2)^2=x_1{}^2+x_2{}^2-2x_1x_2=(x_1+x_2)^2-4x_1x_2=a_1{}^2-4a_0a_2/a_0{}^2.$$

To find the difference of the roots from the last equation we have to extract the square root of the last member. This may be impossible in the domain of the coefficients. If this domain forms a group with respect to the extraction of the square root it is clearly possible in this domain. We know that the system of ordinary complex numbers forms a group with respect to the extraction of any root. Hence we see that, if a_0, a_1, a_2 lie in the domain formed by the ordinary complex numbers, the difference of the roots of A as well as the sum of these roots must lie in the same domain.

The roots themselves may be found from these two functions by means of addition and subtraction. As any domain includes all the quantities resulting by applying these operations to any of its quantities the roots of A must also lie in the given domain of rationality. The roots may also be found by observing that their general linear function

$$\alpha x_1+\beta x_2$$

is rationally expressible as follows :†

$$\alpha x_1+\beta x_2\equiv\tfrac{1}{2}(\alpha+\beta)(x_1+x_2)+\tfrac{1}{2}(\alpha-\beta)(x_1-x_2)$$

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*By enlarging a domain of rationality by the addition of a quantity is meant the forming of the smallest domain that contains the given domain and the added quantity.

†This is an illustration of the general theorem that any rational function of the roots of an algebraic equation of degree n is rationally expressible in terms of a $n!$ valued function of the n roots.

$$\equiv \tfrac{1}{2}-(\alpha+\beta)(a_1/a_0)+(\alpha+\beta)/2a_0\sqrt{a_1^2-4a_0a_2}.$$

By letting $\alpha=1$, $\beta=0$, and $\alpha=0$, $\beta=1$ in this identity we obtain the values of x_1 and x_2 respectively.

As the ordinary complex numbers do not only form a domain of rationality but also a group* with respect to what is frequently called the most general algebraic operation, viz: that represented by

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots a_n=0$$

(a_0, a_1, a_2, ..., a_n being ordinary complex numbers and n any posiive integer), and as they obey the commutative, distributive and associative laws of operation just like real numbers and also the law that a product cannot be zero unless one of the factors is zero, it is clear that we can reason quite generally in regard to symbols representing such numbers. It is probably largely due to this fact that other number systems are not more generally employed. In fact, no really different number system was developed until 1843. In this year Sir William Hamilton discovered and communicated to the Royal Irish Academy the system known as *Quaternions*, which is perhaps still the most important system besides that of the ordinary complex numbers. In the following year Grassmann published his *Ausdehnungslehren* in which he used a number system of a somewhat different form.

Among the investigations of later years those of Weierstrass have probably received the most attention† although important developments have been made in other directions. The fact that the ordinary complex numbers correspond to the points of a plane very naturally led to the thought that a system of higher complex numbers of the form

$$\alpha+\beta i+\gamma j \qquad (\alpha,\ \beta,\ \gamma \text{ being any real numbers})$$

might correspond to the points of space. It was easy to show that the product of two such numbers, multiplied according to the rules of ordinary numbers, may be zero when neither of the factors is zero.‡ This result naturally led to the study of numbers which do not obey all the laws of operations which the ordinary numbers obey.

The main purpose of the preceding remarks was to obtain a fairly clear view of number and of the domain of rationality as these two concepts are fundamental in the study of the solution of algebraic equations. Incidentally we indicated several methods of solving the quadratic equation A. We proceed now to consider some of the other methods of solving this equation. We shall not aim at a complete enumeration of the methods by which A may be solved. In

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*It seems that Poincara was the first who considered the general number systems directly as groups. Cf. Comptes Rendus, t. 99, p. 740.

†*Gottinger Nachrichten*, 1884, page 395.

‡Cf. Harkness and Morley, *Theory of Functions*, page 8.

fact, if we would consider each modification of the operations of finding the roots of A as a new method the number of these methods would clearly be infinite. We may, for instance, form an infinite number of quantics of the form of a quadratic each of which contains the first member of A as a factor. For A may be written in the form

$$a_0x^2+a^2=a_1x.$$

Squaring both members and combining we have

$$ax^4+bx^2+c=0,$$

(a, b, c belonging to the same domain as a_0, a_1, a_2). Since the result is of the same form as A we may repeat the operation any number of times. Hence A is a factor of the quantic

$$A_0x^{2^a}+A_1x^{2^{a-1}}+A_2,$$

(A_0, A_1, A_2 belonging the same domain as a_0, a_1, a_2 and a being any positive integer). The roots of any one of the equations obtained by making these quantics equal zero include the roots of A. As the roots of

$$A_0y^2+A_1y+A_2=0$$

are the 2^{a-1} powers of the roots of A it is clear that none of these transformations can simplify the solution of A. By elimination we may clearly obtain an indefinite number of additional equations containing the roots of A from the given system. In particular, if we eliminate the constant from the biquadratic equation by means of A we obtain a biquadratic equation which has the roots of A and two zero roots. Upon this elimination depends a solution recently publishin this journal. The same result might be obtained by multiplying both members of A by x^2. It is, in general, not well to raise the degree of A in the process of solution since this introduces additional roots and therefore makes the operation more complex.

Perhaps the best known method of solving A is that by which its first member is made a perfect square by the addition of the same quantity to each member. To make the quantic

$$a_0x^2+a_1x+a_2$$

a perfect square without altering its degree we may add to it the quantic

$$ax^2+bx+c$$

where two of the three numbers a, b, c are entirely arbitrary since it is only necessary that the discriminant vanishes. This idea is frequently expressed by say-

ing that the quantic to be added can be chosen in a doubly inflnite number of ways. Since this quantic must also be a perfect square its own discriminant must also vanish. As this imposes another condition on its coefficients we can select the trinomial to be added to both members of A in only a simply infinite number of ways.

This number of choices might at first appear too small since in the ordinary method by which we add a constant to both members of A we apparently select both a and b arbitrarily since we let both equal zero. This would imply a doubly infinite number of choices. This apparent contradiction is explained by the fact that the vanishing of the discriminant of the added trinomial, i. e., the equation

$$b^2=4ac$$

indicates that at least two of the coefficients, including b, must be zero when one is zero. Hence the ordinary method implies that one of the coefficients of the added trinomial is selected arbitrarily and the other in accord with this equation.

To illustrate we inquire what quantics may be added to both members of the special equation

$$x^2-4x+1=0$$

so as to make both members perfect squares. Adding the given general quantic we have the equations

$$(a+1)x^2+(b-4)x+c+1=ax^2+bx+c.$$

Since the discriminants of both members must vanish we have

$$(b-4)^2=4(a+1)(c+1) \text{ and } b^2=4ac.$$

If we assign to b the arbitrary number 2 and eliminate c we have

$$a^2+a+1=0.$$

Hence a and c are the imaginary cube roots of unity, ω_1 and ω_2, and the given equation becomes*

$$-\omega_1{}^2x^2-2x-\omega_2{}^2=\omega_1x^2+2x+\omega_2$$

or

$$-1(\omega_1{}^2x^2+2x+\omega_2{}^2)=\omega_2{}^2x^2+2x+\omega_1{}^2.$$

Extracting the square root from both members we have

$$\pm i(\omega_1x+\omega_2)=\omega_2x+\omega_1$$

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*It should be observed that the product of the two imaginary cube roots of unity is unity and that the square of one is equal to the other.

or $$x=\frac{\omega_1 \mp i\omega_2}{\pm i\omega_1-\omega_2}=2\pm\sqrt{3}.$$

If we let $b=4$ the first discriminant shows that one of the two factors $a+1$, $c+1$ must vanish. If we suppose that the former vanishes the given equation becomes

$$-3=-x^2+4x-4 \text{ or } x^2-4x+4=3.$$

If we suppose that the latter of the given factors vanish we obtain the equation

$$4x^2-4x+1=3x^2.$$

Instead of assigning an arbitrary value to b we might clearly assign an arbitrary value to either of the other coefficients. The simplest method is probably that in which a is made equal to zero. By making a and b equal to the corresponding coefficients with the signs changed of the equation which is to be solved and selecting c so as to satisfy the equation

$$b^2=4ac$$

we have another simple rule for completing the square. A number of other fairly convenient rules can easily be derived from what precedes.

That we can assign the given values to a and b follows from the first of the given discriminants. If we assign this value to a we determine the value of b at the same time but if we commence by assigning the given value to b neither a nor c are fully determined. We still say that the number of choices is simply infinite since a finite number multiplied into a simply infinite number is said to give a simply infinite product. The preceding remarks apply evidently also to the slight modification of the given method which consists in writing A in the form

$$a^2-b^2=0 \text{ instead of } a^2=b^2$$

and factoring the first member according to the well known formula

$$a^2-b^2=(a+b)(a-b)=(-b-a)(b-a)$$

instead of extracting the square root of the two members.

Another simple method of solving A may be described as follows: The equation A is satisfied by the affixes of two points and gives the elementary symmetric functions of these affixes. As all rational symmetric functions can be expressed rationally in terms of the elementary symmetric functions we know the affix of the middle point of the join of the roots. If the points of the plane are so transformed that this point becomes the origin the roots are the affixes of the extremities of a diameter of a circle whose center is the origin. Hence the equa-

tion in the new variable must be a pure quadratic and the solution is readily completed. If we do not assume that the coefficients are real, one root may be real while the other is imaginary. In fact the roots may be the affixes of any two points.

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from January Number.]

Proposition XXV. *If two straights (Fig. 30.) AX, BX existing in the same plane (standing upon AB, one indeed at an acute angle in the point A, and the other perpendicular at the point B) so always approach more to each other mutually, toward the parts of the point X, that nevertheless their distance is always greater than a certain assigned length, the hypothesis of acute angle is destroyed.*

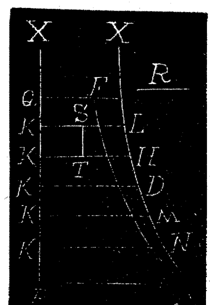

Fig. 30.

Proof. Let R be the assigned length. If therefore in BX is assumed a certain BK any chosen multiple of the proposed length R; it follows (from the preceding Scholion) that the perpendicular erected from the point K toward the parts of AX will meet it at some point L; and again (from the present hypothesis) it follows that this KL will be greater than the aforesaid length R. Furthermore BK is understood divided into portions KK, each equal to R, even until KB is itself equal to the length R. Finally from the points K are erected to BX perpendiculars meeting AX in points L, H, D, M, even to the point N nearest the point A. Now I proceed thus.

The four angles together of the quadrilateral $KHLK$, more remote from the base AB, will be (from the preceding Proposition) greater than the four angles together of the quadrilateral $KDHK$, nearer to this base; of which quadrilateral in the same way the four angles together will be greater than the four angles together of the quadrilateral $KMDK$ subsequent toward this base. And so always even to the last quadrilateral $KNAB$, whose four angles together assuredly will be the least, in reference to the four angles together of each of the quadrilaterals ascending toward the points X.

But since are present as many quadrilaterals described in the aforesaid manner, as are, except the base AB, perpendiculars let fall from points of AX to